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Lower estimates for the eigenfunctions of the Schrödinger operator

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Let $G=(a, b)$ be a bounded interval, $q\in L^1(G)$ a complex function and consider the formal differential operator

$$Lu = -u''+q\cdot u.$$

A function $u_i\colon G\to \mathbf{C}, u_i\not\equiv 0\ (i=0; 1, \dots)$ is said to be an eigenfunction of order i (of the operator L) with the eigenvalue $\lambda\in\mathbf{C}$, if it is absolutely continuous together with its derivative on every compact subinterval of G, and for almost all $x\in G$ the equation

$$-u_i''(x)+q(x)\cdot u_i(x) = \lambda\cdot u_i(x)+u_{i-1}(x) \tag{1}$$

holds, where $u_{i-1}\equiv 0$ for $i=0$ and u_{i-1} is an eigenfunction of order $i-1$, with the eigenvalue λ, for $i\geqq 1$.

It is known (see [1], pp. 167—169) that in this case u_i, together with its derivative, can be continuously extended to the closed interval $[a, b]$, and the extended functions are absolutely continuous on the whole interval $[a, b]$. Hence $u_i\in L^p(G)$ for all $1\leqq p\leqq\infty$. For the sake of brevity, we shall use the notation $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(G)}$.

The aim of the present paper is to prove the following.

Theorem. *Let $G=(a, b)$ be a bounded interval and $q\in L^1(G)$ a complex function. Then, for an arbitrary eigenfunction u_i of order $i\geqq 0$ with the eigenvalue λ, and for any $1\leqq p<q<\infty$, the following estimates hold:*

$$\frac{\|u_i\|_\infty}{\|u_i\|_p}\geqq C_1\cdot(1+|\operatorname{Im}\sqrt{\lambda}|)^{1/p}, \tag{2}$$

$$C_2\cdot(1+|\operatorname{Im}\sqrt{\lambda}|)^{1/p-1/q}\leqq\frac{\|u_i\|_q}{\|u_i\|_p}\leqq C_3\cdot(1+|\operatorname{Im}\sqrt{\lambda}|)^{1/p-1/q}, \tag{3}$$

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where the positive constants C_1, C_2, C_3 depend on i, $b-a$, $\|q\|_1$, but do not depend on λ, p and q: $C_j=C_j(i, b-a, \|q\|_1)$, $j=1, 2, 3$.

Remark. The estimate

$$\frac{\|u_i\|_\infty}{\|u_i\|_p} \leqq C_4(i, b-a, \|q\|_1)\cdot(1+|\operatorname{Im}\sqrt{\lambda}|)^{1/p} \tag{4}$$

is also true; this was proved by I. Joó [5]. Thus our result is exact from the view point of dependence on λ.

In the proof of the theorem we shall use the following result of [5]: If u_i is an arbitrary eigenfunction of order $i\geqq 1$ with the eigenvalue λ and $u_{i-1}\equiv Lu_i-\lambda\cdot u_i$ then

$$\|u_{i-1}\|_\infty \leqq C_0(i)\cdot\left(1+|\sqrt{\lambda}|\right)\cdot\left(1+|\operatorname{Im}\sqrt{\lambda}|\right)\cdot\|u_i\|_\infty, \tag{5}$$

where the constant $C_0(i)=C_0(i, b-a, \|q\|_1)$ does not depend on λ.

We recall the formula of E. C. Titchmarsh [2], having been extended for eigenfunctions of higher order in [5]: for any $x-t, x+t\in G$ and $i\in\{0, 1, \dots\}$,

$$u_i(x+t)+u_i(x-t) = 2\cdot u_i(x)\cdot\cos\left(\sqrt{\lambda}\,t\right)+ \\ +\int_{x-t}^{x+t}\left(q(\xi)\cdot u_i(\xi)-u_{i-1}(\xi)\right)\cdot\frac{\sin\sqrt{\lambda}\,(t-|x-\xi|)}{\sqrt{\lambda}}\,d\xi, \tag{6}$$

if $\lambda\neq 0$.

We mention also the simple inequalities:

$$\exp(|\operatorname{Im} z|)-1 \leqq |2\cdot\cos z|, \quad |2\cdot\sin z| \leqq \exp(|\operatorname{Im} z|)+1 \quad (z\in\mathbf{C}). \tag{7}$$

The proof of the theorem will be based on the following

Proposition. *Let u_i be an arbitrary eigenfunction of order $i\geqq 0$ with the eigenvalue λ. Then, setting for brevity $\nu=\operatorname{Im}\sqrt{\lambda}$ and $d_{a,b}(x)=\min(|x-a|, |x-b|)$, we have*

$$(m_i \equiv)\max_{x\in[a,b]}|u_i(x)|\cdot\left(1+|\nu|\cdot d_{a,b}(x)\right)^{-i}\cdot\exp\left(|\nu|\cdot d_{a,b}(x)\right) \leqq M_i\cdot\|u_i\|_\infty, \tag{8}$$

where the constant $M_i=M_i(b-a, \|q\|_1)$ does not depend on λ.

Proof. We work by induction on i. For $i=-1$ (8) is formally true with $M_{-1}\equiv 0\ (u_{-1}\equiv 0)$. Let now $i\geqq 0$ be arbitrary and suppose (8) is true for $i-1$. In case $|\sqrt{\lambda}|\leqq 1+2^i\cdot\|q\|_1$ we obviously have

$$m_i \leqq \exp\left((1+2^i\cdot\|q\|_1)\cdot(b-a)\right)\cdot\|u_i\|_\infty. \tag{9}$$

Consider now the case $|\sqrt{\lambda}|>1+2^i\cdot\|q\|_1$. Denote $y\in[a, b]$ such a point where the maximum on the left side of (8) is attained. Then

$$m_i = |u_i(y)|\cdot(1+|\nu|\cdot t)^{-i}\cdot\exp(|\nu| t) \quad \left(t\equiv d_{a,b}(y)\right). \tag{10}$$

By properties (10), (7), (6), (5) and the inductive hypothesis we can write the following chain of inequalities:

$$m_i\cdot(1+|v|t)^i-\|u_i\|_\infty \leqq m_i\cdot(1+|v|t)^i-|u_i(y)| = |u_i(y)|\cdot(\exp(|v|t)-1) \leqq$$

$$\leqq \left|2\cdot u_i(y)\cdot\cos(\sqrt{\lambda}\,t)\right| \leqq |u_i(y-t)+u_i(y+t)|+\left|\int_{y-t}^{y+t}(q(\xi)u_i(\xi)-\right.$$

$$\left.-u_{i-1}(\xi))\frac{\sin\sqrt{\lambda}(t-|y-\xi|)}{\sqrt{\lambda}}\,d\xi\right| \leqq 2\cdot\|u_i\|_\infty+\frac{\|q\|_1}{2\cdot|\sqrt{\lambda}|}\cdot\max_{|y-\xi|\leqq t}|u_i(\xi)|\cdot$$

$$\cdot(1+\exp(|v|(t-|y-\xi|)))+\frac{t}{|\sqrt{\lambda}|}\cdot\max_{|y-\xi|\leqq t}|u_{i-1}(\xi)|\cdot(1+\exp(|v|(t-|y-\xi|))) \leqq$$

$$\leqq 2\cdot\|u_i\|_\infty+2^{-i-1}\cdot(m_i\cdot(1+2\cdot|v|\,t)^i+\|u_i\|_\infty)+\frac{t}{|\sqrt{\lambda}|}\cdot(m_{i-1}(1+2\cdot|v|\,t)^{i-1}+\|u_{i-1}\|_\infty) \leqq$$

$$\leqq \frac{5}{2}\cdot\|u_i\|_\infty+\frac{1}{2}m_i\cdot(1+|v|\,t)^i+\frac{t}{|\sqrt{\lambda}|}\cdot(1+M_{i-1}\cdot(1+2\cdot|v|\,t)^{i-1})\cdot C_0(i)\cdot(1+|\sqrt{\lambda}|)\cdot$$

$$\cdot(1+|v|)\cdot\|u_i\|_\infty \leqq \frac{5}{2}\cdot\|u_i\|_\infty+\frac{1}{2}m_i\cdot(1+|v|\,t)^i+\frac{1+|\sqrt{\lambda}|}{|\sqrt{\lambda}|}(1+M_{i-1})\cdot(1+2\cdot|v|\,t)^{i-1}\cdot$$

$$\cdot C_0(i)\cdot(t+|v|\,t)\cdot\|u_i\|_\infty \leqq \frac{1}{2}m_i\cdot(1+|v|\,t)^i+\|u_i\|_\infty\cdot$$

$$\cdot\left(\frac{5}{2}+2^i\cdot(1+M_{i-1})\cdot C_0(i)\cdot(1+b-a)\cdot(1+|v|\,t)^i\right).$$

Hence

$$m_i \leqq (7+2^{i+1}\cdot(1+M_{i-1})\cdot C_0(i)\cdot(1+b-a))\cdot\|u_i\|_\infty. \tag{11}$$

It follows from (9) and (11) that (8) is true for i if we put

$$M_i \equiv \max(\exp((1+2^i\cdot\|q\|_1)\cdot(b-a)),\ 7+2^{i+1}\cdot(1+M_{i-1})\cdot C_0(i)\cdot(1+b-a)).$$

The proposition is proved.

Corollary. *For any* $0<\alpha<1$ *there exists a constant* $M_i(\alpha)=M_i(\alpha, b-a, \|q\|_1)$ *independent of* λ *such that*

$$\max_{x\in[a,b]}|u_i(x)|\exp(\alpha\cdot|v|\cdot d_{a,b}(x)) \leqq M_i(\alpha)\cdot\|u_i\|_\infty. \tag{12}$$

Let us turn to the proof of the theorem. Choosing for instance $\alpha=1/2$, we have by (12) for all $x\in G$:

$$|u_i(x)| \leqq M_i(1/2)\cdot\|u_i\|_\infty\cdot\exp\left(-\frac{1}{2}\cdot|v|\cdot d_{a,b}(x)\right).$$

Taking the $L^p(G)$ norm of both sides, we obtain for $|\nu|\geqq 1$

$$\|u_i\|_p \leqq M_i(1/2)\cdot\|u_i\|_\infty\cdot 4^{1/p}\cdot p^{-1/p}\cdot|\nu|^{-1/p}\leqq$$

$$\leqq 4\cdot M_i(1/2)\cdot\|u_i\|_\infty\cdot|\nu|^{-1/p}\leqq 8\cdot M_i(1/2)\cdot\|u_i\|_\infty\cdot(1+|\nu|)^{-1/p},$$

and hence,

$$\|u_i\|_\infty \geqq \big(8\cdot M_i(1/2)\big)^{-1}\cdot\|u_i\|_p\cdot(1+|\nu|)^{1/p}. \tag{13}$$

On the other hand, in the case $|\nu|<1$ we have obviously

$$\|u_i\|_p \leqq (b-a)^{1/p}\cdot\|u_i\|_\infty \leqq (1+b-a)\cdot\|u_i\|_\infty \leqq 2\cdot(1+b-a)\cdot\|u_i\|_\infty\cdot(1+|\nu|)^{-1/p},$$

and

$$\|u_i\|_\infty \geqq \big(2\cdot(1+b-a)\big)^{-1}\cdot\|u_i\|_p\cdot(1+|\nu|)^{1/p}; \tag{14}$$

and (13) and (14) yield the estimate (2) with

$$C_1\big(i, b-a, \|q\|_1\big) \equiv \min\big((8\cdot M_i(1/2))^{-1}, (2\cdot(1+b-a))^{-1}\big).$$

The estimates (3) are easy consequences of (2) and (4). The theorem is proved.

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